

HARDY-TYPE INEQUALITIES FOR VECTOR FIELDS WITH THE TANGENTIAL COMPONENTS VANISHING

XINGFEI XIANG AND ZHIBING ZHANG

ABSTRACT. This note studies the Hardy-type inequalities for vector fields with the L^1 norm of the curl. In contrast to the well-known results in the whole space for the divergence-free vectors, we generalize the Hardy-type inequalities to the bounded domains and to the non-divergence-free vector fields with the tangential components on the boundary vanishing.

1. INTRODUCTION

This note is devoted to establish the Hardy-type inequality for vector fields in L^1 space in 3-dimensional bounded domains. We prove that for the vector field \mathbf{u} with the tangential components on the boundary vanishing, the L^1 norm of $\mathbf{u}/|x|$ can be controlled by the L^1 norm of $(1 + \ln|x|) \operatorname{div} \mathbf{u}$ and the L^1 norm of $\operatorname{curl} \mathbf{u}$.

This work belongs to the field of the L^1 estimate for vector fields. Starting with the pioneering work by Bourgain and Brezis in [2], the L^1 estimate has been well studied by many mathematicians, see [2-7, 10-12, 15, 16, 18] and the references therein. In particular, Bourgain and Brezis in [4] obtained the delicate $L^{3/2}$ estimate for the divergence-free vectors on the torus \mathbb{T}^3 . Maz'ya in [11] (also see Bousquet and Van Schaftingen's more general result in [6] by introducing the cancellation condition) obtained a Hardy-type inequality for the divergence-free vector fields \mathbf{u} (not direct but implied)

$$\left\| \frac{\mathbf{u}}{|x|} \right\|_{L^1(\mathbb{R}^3)} \leq C \|\operatorname{curl} \mathbf{u}\|_{L^1(\mathbb{R}^3)}. \quad (1.1)$$

This actually gives the essential answer to the problem raised by Bourgain and Brezis in [4, open problem 1]. Bousquet and Mironescu give an elementary proof of (1.1) in [5].

In this note we consider the problem in bounded domains, in particular with the singularity (the origin) being on the boundary. For the case of the singularity being in the interior of the domain, we can easily get the similar estimate (1.1) in bounded domains by taking the cut-off method. However, if the singularity is on the boundary, the usual flattening boundary and the localization by partition of unity does not work. The main reason is that by taking the flattening boundary there would arise the L^1 norm of $\nabla \mathbf{u}$, this term can't be controlled by the L^1 norm of $\operatorname{curl} \mathbf{u}$ for the vector \mathbf{u} being divergence-free, and hence a new approach should be considered.

To get around this difficulty we apply the Helmholtz-Weyl decomposition for vector fields in bounded domains (see [9, Theorem 2.1]):

$$\mathbf{u} = \operatorname{curl} \mathbf{w}_{\mathbf{u}} + \nabla p_{\mathbf{u}} + \mathcal{H}_{\mathbf{u}},$$

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where $\mathcal{H}_{\mathbf{u}}$ is the harmonic part depending only on the domain. Our strategy is to get the estimates for the curl part $\text{curl } \mathbf{w}_{\mathbf{u}}$ and the gradient part $\nabla p_{\mathbf{u}}$ respectively. Thanks to Solonnikov's work in [13, 14] (also see Beirão da Veiga and Berselli's work in [1, p.606]), the vector $\mathbf{w}_{\mathbf{u}}$ in the curl part satisfies Petrovsky type elliptic system, and hence there exists a single Green's matrix $\mathcal{G}(x, y)$ such that

$$\mathbf{w}(x) = \int_{\Omega} \mathcal{G}(x, y) \text{curl } \mathbf{u}(y) dy.$$

Based on the estimate for the Green's matrix, we can obtain the estimate on the curl part. For the gradient part $\nabla p_{\mathbf{u}}$ we can use the classical elliptic theory to get the estimate.

Before stating the main result, we make the following assumption on the domain:

(O) Ω is a bounded in \mathbb{R}^3 with smooth boundary, in all cases considered here the class C^2 will be sufficient. The second Betti number is 0 which is understood as there is no holes in the domain.

Denote by $\nu(x)$ the unit outer normal vector at $x \in \partial\Omega$. The main result now reads:

Theorem 1.1. *Assume that the domain Ω satisfies the assumption (O). Then for any $\mathbf{u} \in C^1(\bar{\Omega}, \mathbb{R}^3)$ with $\nu \times \mathbf{u} = 0$ on $\partial\Omega$, we have*

$$\left\| \frac{\mathbf{u}}{|x|} \right\|_{L^1(\Omega)} \leq C \left(\|\ln |x| \text{div } \mathbf{u}\|_{L^1(\Omega)} + \|\text{curl } \mathbf{u}\|_{L^1(\Omega)} \right), \quad (1.2)$$

where the constant C depends only on the domain Ω .

Remark 1.2. *We need to mention that*

- (i) *The proof method of this theorem is not applicable for the vector fields with the normal components on the boundary vanishing. The reason is that the key step we used is the zero extension of $\text{curl } \mathbf{u}$ outside of the domain, but this does not hold for the vector fields with the normal components on the boundary vanishing.*
- (ii) *By a similar discussion, one can get the estimate for elliptic system associating with the Hardy-type inequality. Let $\mathbf{f} \in C^1(\bar{\Omega}, \mathbb{R}^3)$ with $\text{div } \mathbf{f} = 0$ in Ω and $\nu \cdot \mathbf{f} = 0$ on the boundary. Then for the elliptic system $\mathcal{L}\mathbf{u} = \mathbf{f}$ with the form of the solution can be expressed by*

$$\mathbf{u} = \int_{\Omega} \mathcal{G}(x, y) \mathbf{f}(y) dy,$$

where the Green's matrix $\mathcal{G}(x, y)$ satisfy the inequality (2.1), we have

$$\left\| \frac{\mathbf{u}}{|x|^2} \right\|_{L^1(\Omega)} \leq C \|\mathbf{f}\|_{L^1(\Omega)}.$$

However, this may not be true for the single elliptic equation. The typical example is $\mathcal{L} = \Delta$ with zero boundary condition.

The organization of this paper is as follows. In Section 2, we will give the proof of Theorem 1.1. In Section 3, the Hardy-type inequality in L^p space with $1 < p < 3/2$ will be treated. We will show that the L^p norm of $\mathbf{u}/|x|$ can be controlled by the L^p norm of the $\text{div } \mathbf{u}$ and the $\text{curl } \mathbf{u}$ whether or not the singularity is on the boundary.

Throughout the paper, the bold typeface is used to indicate vector quantities; normal typeface will be used for vector components and for scalars.

2. HARDY-TYPE INEQUALITIES FOR VECTOR FIELDS WITH L^1 DATA

The key step in the proof of the main theorem is the estimate for the singular integral involving the operator curl. This estimate was first obtained by Maz'ya in [11] in the case of the kernel being the Newtonian potential and of the domain being the entire space. The case where the kernel being the Green's function associating with the elliptic operator in the entire space was considered by Bousquet and Van Schaftingen (see [6, Lemma 2.2]). We generalized their kernel to a more general case. The method of our proof goes back to work by Bousquet and Mironescu in [5].

Lemma 2.1. *Suppose that the function $A(x, y) \in C^1(\Omega \times \mathbb{R}^3)$ for $x \neq y$ satisfying*

$$(i) \quad |A(x, y)| \leq \frac{C}{|x - y|^2}; \quad (ii) \quad |\nabla_y A(x, y)| \leq \frac{C}{|x - y|^3}. \quad (2.1)$$

Let $\Psi \in L^1(\mathbb{R}^3, \mathbb{R}^3)$ with $\operatorname{div} \Psi = 0$. Then there exists a constant C such that

$$\left\| \frac{1}{|x|} \int_{\mathbb{R}^3} A(x, y) \Psi(y) dy \right\|_{L^1(\Omega)} \leq C \|\Psi\|_{L^1(\mathbb{R}^3)}. \quad (2.2)$$

Proof. For simplicity of the notations, we let

$$I_\Sigma := \int_{\Sigma} A(x, y) \Psi(y) dy.$$

Then write

$$\int_{\mathbb{R}^3} A(x, y) \Psi(y) dy = I_{\{|y| > 2|x|\}} + I_{\{\frac{|x|}{4} < |y| < 2|x|\}} + I_{\{|y| < \frac{|x|}{4}\}}. \quad (2.3)$$

The estimation of these integrals are achieved as follows. For all x, y satisfying $2|x| < |y|$, the inequality $|x - y| \geq |y|/2$ holds, then using Fubini's theorem, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{1}{|x|} |I_{\{|y| > 2|x|\}}| dx &\leq \int_{\mathbb{R}^3} |\Psi(y)| \int_{\{|x| < \frac{|y|}{2}\} \cap \Omega} |A(x, y)| \frac{1}{|x|} dx dy \\ &\leq C \int_{\mathbb{R}^3} |\Psi(y)| \frac{1}{|y|^2} \int_{\{|x| < \frac{|y|}{2}\}} \frac{1}{|x|} dx dy. \end{aligned} \quad (2.4)$$

It is easy to see that the last term of the above inequality can be controlled by the L^1 norm of $\Psi(x)$. We now estimate the second term in the right side of (2.3). Using Fubini's theorem again, it follows that

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{1}{|x|} |I_{\{\frac{|x|}{4} < |y| < 2|x|\}}| dx &\leq \int_{\mathbb{R}^3} |\Psi(y)| \int_{\{\frac{|y|}{2} < |x| < 4|y|\}} \frac{1}{|x||x - y|^2} dx dy \\ &\leq C \int_{\mathbb{R}^3} |\Psi(y)| \int_{\{|x - y| < 5|y|\}} \frac{1}{|y||x - y|^2} dx dy, \end{aligned} \quad (2.5)$$

the last term in the above inequality can also be controlled by $\|\Psi(x)\|_{L^1(\mathbb{R}^3)}$.

We now estimate the integral involving the term $I_{\{|y| < |x|/4\}}$. Take the cut-off function $\eta(t)$ such that $\eta(t) = 0$ for $t > 1/2$, $\eta(t) = 1$ for $0 < t < 1/4$, and $|\eta'(t)| \leq 8$. Applying the equality

$$\int_{\mathbb{R}^3} \operatorname{div}(y_i A(x, y) \eta(|y|/|x|) \Psi(y)) dy = 0 \quad \text{for } i = 1, 2, 3,$$

and then using $\operatorname{div} \Psi = 0$ we have

$$\begin{aligned} & \int_{\mathbb{R}^3} A(x, y) \eta(|y|/|x|) \Psi(y) dy \\ &= - \int_{\mathbb{R}^3} \nabla_y (A(x, y) \eta(|y|/|x|)) \cdot \Psi(y) (y_1, y_2, y_3) dy. \end{aligned}$$

The conditions (i) and (ii), for $|y| < |x|/2$, imply that

$$|\nabla_y (A(x, y) \eta(|y|/|x|))| \leq C \left(\frac{1}{|x - y|^3} + \frac{1}{|x||x - y|^2} \right) \leq C \frac{1}{|x|^3}.$$

This shows that

$$\int_{\Omega} \frac{1}{|x|} \left| \int_{\mathbb{R}^3} A(x, y) \eta(|y|/|x|) \Psi(y) dy \right| dx \leq C \int_{\mathbb{R}^3} |\Psi(y)| \int_{\{|x| > 2|y|\}} \frac{|y|}{|x|^4} dx dy. \quad (2.6)$$

The last term in the above inequality can be controlled by $\|\Psi(x)\|_{L^1(\mathbb{R}^3)}$. Noting that

$$\begin{aligned} & \int_{\mathbb{R}^3} A(x, y) \eta(|y|/|x|) \Psi(y) dy \\ &= I_{\{|y| < \frac{|x|}{4}\}} + \int_{\{\frac{|x|}{4} < |y| < \frac{|x|}{2}\}} A(x, y) \eta(|y|/|x|) \Psi(y) dy. \end{aligned}$$

The estimation involving the last term of the above equality can be obtained by (2.5). Thus from (2.6) it follows that

$$\int_{\Omega} \frac{1}{|x|} \left| I_{\{|y| < \frac{|x|}{4}\}} \right| dx \leq C \|\Psi(x)\|_{L^1(\mathbb{R}^3)}.$$

Plugging (2.4), (2.5) and the above inequality back to (2.3) we obtain (2.2). We finish our proof. \square

Remark 2.2. *It is easy to see that this lemma is still true if replaced the scalar function $A(x, y)$ by a matrix $\mathcal{G}(x, y)$ satisfying the inequalities in (2.1).*

By a similar discussion of Lemma 2.1 and applying Fubini's theorem, we can get the estimate of the singular integral for scalar functions.

Lemma 2.3. *Suppose that the function $\ln|x|\Psi \in L^1(\Omega)$ and assume that the function $A(x, y) \in C^1(\bar{\Omega} \times \bar{\Omega})$ for $x \neq y$ satisfying the inequalities in (2.1). Then there exists a constant C depending only on the domain such that*

$$\left\| \frac{1}{|x|} \int_{\Omega} A(x, y) \Psi(y) dy \right\|_{L^1(\Omega)} \leq C \|(1 + \ln|x|)\Psi\|_{L^1(\Omega)}. \quad (2.7)$$

For the vector $\Psi = \operatorname{curl} \Phi$ with $\nu \times \Phi = 0$ on the boundary, the estimate in Lemma 2.3 can be improved.

Lemma 2.4. *Let $\Phi \in C^1(\bar{\Omega}, \mathbb{R}^3)$ with $\nu \times \Phi = 0$ on the boundary. Suppose that the function $A(x, y) \in C^1(\bar{\Omega} \times \bar{\Omega})$ for $x \neq y$ satisfying the inequalities in (2.1), then there exists a constant C such that*

$$\left\| \frac{1}{|x|} \int_{\Omega} A(x, y) \operatorname{curl} \Phi dy \right\|_{L^1(\Omega)} \leq C \|\operatorname{curl} \Phi\|_{L^1(\Omega)}. \quad (2.8)$$

Proof. Let $\tilde{\Phi}$ be the zero extension of the vector Φ outside of the Ω . Then $\operatorname{curl} \tilde{\Phi} = 0$ in \mathbb{R}^3 in the sense of distribution, and we get

$$\left\| \frac{1}{|x|} \int_{\Omega} A(x, y) \operatorname{curl} \Phi dy \right\|_{L^1(\Omega)} = \left\| \frac{1}{|x|} \int_{\mathbb{R}^3} A(x, y) \operatorname{curl} \tilde{\Phi} dy \right\|_{L^1(\Omega)}.$$

From Lemma 2.1, it follows that

$$\left\| \frac{1}{|x|} \int_{\Omega} A(x, y) \operatorname{curl} \Phi dy \right\|_{L^1(\Omega)} \leq C \|\operatorname{curl} \tilde{\Phi}\|_{L^1(\mathbb{R}^3)} \leq C \|\operatorname{curl} \Phi\|_{L^1(\Omega)}.$$

□

We now give the proof of the main theorem.

Proof of Theorem 1.1. From the Helmholtz-Weyl decomposition (see [9, Theorem 2.1]), for every $\mathbf{u} \in C^1(\bar{\Omega})$ there exists a decomposition

$$\mathbf{u} = \nabla p_{\mathbf{u}} + \operatorname{curl} \mathbf{w}_{\mathbf{u}}, \quad (2.9)$$

where the function $p_{\mathbf{u}} \in W^{2,p}(\Omega)$ satisfying

$$\Delta p_{\mathbf{u}} = \operatorname{div} \mathbf{u} \quad \text{in } \Omega, \quad p_{\mathbf{u}} = 0 \quad \text{on } \partial\Omega; \quad (2.10)$$

the vector $\mathbf{w}_{\mathbf{u}} \in X_n^{2,p}(\Omega)$ with $X_n^{2,p}$ defined by

$$X_n^{2,p}(\Omega) \equiv \{ \mathbf{w} \in W^{2,p}(\Omega) : \operatorname{div} \mathbf{w} = 0, \nu \cdot \mathbf{w} = 0 \text{ on } \partial\Omega \}$$

and the vector $\mathbf{w}_{\mathbf{u}}$ satisfying the elliptic system

$$\begin{cases} \operatorname{curl} \operatorname{curl} \mathbf{w}_u = \operatorname{curl} \mathbf{u} & \text{in } \Omega, \\ \operatorname{div} \mathbf{w}_u = 0 & \text{in } \Omega, \\ \nu \times \operatorname{curl} \mathbf{w}_u = \nu \times \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \nu \cdot \mathbf{w}_u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.11)$$

By the classical elliptic equation theory, we see that the solution $p_{\mathbf{u}}$ of the equation (2.10) has the form

$$p_{\mathbf{u}} = \int_{\Omega} G_1(x, y) \operatorname{div} \mathbf{u}(y) dy$$

and Green's function $G_1(x, y)$ satisfies the inequality (2.1). Lemma 2.3 gives

$$\left\| \frac{\nabla p_{\mathbf{u}}}{|x|} \right\|_{L^1(\Omega)} \leq C \|(1 + \ln |x|) \operatorname{div} \mathbf{u}\|_{L^1(\Omega)}. \quad (2.12)$$

We now estimate the term involving the operator curl. Note that the elliptic system (2.11) is of Petrovsky type (see Solonnikov [13, 14], also see the reference [1, p.606] by Beirão da Veiga and Berselli). Therefore, the solution of (2.11) can be expressed by

$$\mathbf{w}(x) = \int_{\Omega} \mathcal{G}(x, y) \operatorname{curl} \mathbf{u}(y) dy.$$

where the Green's matrix $\mathcal{G}(x, y)$ is written as

$$\mathcal{G}(x, y) = \mathbf{G}_2(x, y) + \mathbf{R}(x, y).$$

The leading term $\mathbf{G}_2(x, y)$ satisfies the estimate (see [1, p.608])

$$|D_x^\alpha D_y^\beta \mathbf{G}_2(x, y)| \leq \frac{C(\alpha, \beta, \Omega)}{|x - y|^{\alpha+\beta+1}} \quad (2.13)$$

and the matrix $\mathbf{R}(x, y)$ satisfies (see [1, p.610])

$$|D_x^\alpha D_y^\beta \mathbf{R}(x, y)| \leq \frac{C(\alpha, \beta, \Omega)}{|x - y|^{\alpha+\beta+1+\gamma}} \quad \text{with } \gamma > 0. \quad (2.14)$$

Note that

$$\operatorname{curl} \mathbf{w}(x) = \int_{\Omega} \operatorname{curl}_x (G^1, G^2, G^3)(x, y) \operatorname{curl} \mathbf{u}(y) dy$$

where $(G^1, G^2, G^3) = \mathcal{G}(x, y)^T$ is the row vector of the matrix $\mathcal{G}(x, y)$. Then by the estimates (2.13) and (2.14), Lemma 2.8 is applicable for $\operatorname{curl} \mathbf{w}(x)$, and we have

$$\left\| \frac{\operatorname{curl} \mathbf{w}_u}{|x|} \right\|_{L^1(\Omega)} \leq C \|\operatorname{curl} \mathbf{u}\|_{L^1(\Omega)}. \quad (2.15)$$

Combing the estimates (2.12) and (2.15), we complete the proof. \square

3. HARDY-TYPE INEQUALITIES FOR VECTOR FIELDS WITH L^p DATA

To show the Hardy-type inequalities for vector fields with L^p data, we first give the estimate on the vector field itself by the L^1 norm of the operators div and curl.

Lemma 3.1. *Assume that the domain Ω satisfies the assumption (O), and let $\mathbf{u} \in C^1(\bar{\Omega})$ with $\nu \times \mathbf{u} = 0$ on $\partial\Omega$. Then for any $1 \leq p < 3/2$ we have*

$$\|\mathbf{u}\|_{L^p(\Omega)} \leq C(p, \Omega) (\|\operatorname{div} \mathbf{u}\|_{L^1(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{L^1(\Omega)}), \quad (3.1)$$

where the constant C depends only on p and the domain Ω .

Proof. From the fundamental theorem of vector calculus, it follows that

$$4\pi \mathbf{u} = -\nabla \int_{\Omega} \frac{1}{|x - y|} \operatorname{div} \mathbf{u}(y) dy + \operatorname{curl} \int_{\Omega} \frac{1}{|x - y|} \operatorname{curl} \mathbf{u}(y) dy + \nabla v, \quad (3.2)$$

where the function v is defined by

$$v(x) = \int_{\partial\Omega} \frac{1}{|x - y|} \nu \cdot \mathbf{u}(y) dS_y.$$

We estimate each term in (3.2). The L^p estimates on the first and the second terms in the right side of (3.2) can be obtained by applying the Minkowski's integral inequality.

Therefore, it is necessary to show the estimate on $v(x)$. By Lemma A.1 in Appendix in [18] and by Claim 1 in the proof of Theorem 1.1 in [18], for $1 < p < 3/2$ we have

$$\|\nabla v(x)\|_{L^p(\Omega)} \leq C(p, \Omega) \|\nu \cdot \mathbf{u}\|_{W^{-1/p, p}(\partial\Omega)} \leq C(p, \Omega) \|J\|_{W^{-1/p, p}(\partial\Omega)}, \quad (3.3)$$

where J is defined by

$$J = 2\pi \left(\nu, \operatorname{grad} \int_{\Omega} \frac{1}{r} \operatorname{div} \mathbf{u} dy - \operatorname{curl} \int_{\Omega} \frac{1}{r} \operatorname{curl} \mathbf{u} dy \right).$$

We now estimate J , and let

$$w = \int_{\Omega} \frac{1}{|x - y|} \operatorname{div} \mathbf{u} dy.$$

Green's formula and the trace theorem in $W^{1, q}(\Omega)$ yield that

$$\left\| \frac{\partial w}{\partial \nu} \right\|_{W^{-1/p, p}(\partial\Omega)} \leq C(p, \Omega) \left(\|\nabla w\|_{L^p(\Omega)} + \|\operatorname{div} \mathbf{u}\|_{(W^{1, q}(\Omega))^*} \right), \quad (3.4)$$

where q is the exponent conjugate to p , $(\cdot)^*$ denotes the dual space of (\cdot) . Using the Minkowski's integral inequality again, we get

$$\|\nabla w\|_{L^p(\Omega)} \leq C(p, \Omega) \|\operatorname{div} \mathbf{u}\|_{L^1(\Omega)}. \quad (3.5)$$

Since $W^{1, r}$ is continuously embedded into L^∞ for $r > 3$, L^1 is continuously embedded into $(W^{1, r}(\Omega))^*$. Hence, for any $1 < p < 3/2$, (3.4) and (3.5) yield that

$$\left\| \frac{\partial}{\partial \nu} \int_{\Omega} \frac{1}{|x - y|} \operatorname{div} \mathbf{u} \right\|_{W^{-1/p, p}(\partial\Omega)} \leq C \|\operatorname{div} \mathbf{u}\|_{L^1(\Omega)}. \quad (3.6)$$

The trace theorem ([8, Lemma 1]) in the space $\{\mathbf{u} \in L^p(\Omega), \operatorname{div} \mathbf{u} \in L^p(\Omega)\}$ gives

$$\left\| \left(\nu, \operatorname{curl} \int_{\Omega} \frac{1}{r} \operatorname{curl} \mathbf{u} dy \right) \right\|_{W^{-1/p, p}(\partial\Omega)} \leq C(p, \Omega) \left\| \operatorname{curl} \int_{\Omega} \frac{1}{r} \operatorname{curl} \mathbf{u} dy \right\|_{L^p(\Omega)}.$$

Thus the Minkowski's integral inequality, for $1 < p < 3/2$, implies that

$$\left\| \left(\nu, \operatorname{curl} \int_{\Omega} \frac{1}{r} \operatorname{curl} \mathbf{u} dy \right) \right\|_{W^{-1/p, p}(\partial\Omega)} \leq C(p, \Omega) \|\operatorname{curl} \mathbf{u}\|_{L^1(\Omega)}. \quad (3.7)$$

From (3.6) and (3.7), we obtain the estimate on J ,

$$\|J\|_{W^{-1/p, p}(\partial\Omega)} \leq C(p, \Omega) (\|\operatorname{div} \mathbf{u}\|_{L^1(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{L^1(\Omega)}). \quad (3.8)$$

Thus from (3.3) and (3.8) we obtain (3.1) if $1 < p < 3/2$. For $p = 1$ we apply Hölder's inequality in (3.1), and hence we finish our proof. \square

We now show the Hardy-type inequalities with L^p data for the vectors.

Theorem 3.2. *Assume that the domain Ω satisfies the assumption (O) with the origin $\mathbf{0} \in \bar{\Omega}$. Let $\mathbf{u} \in W^{1, p}(\Omega, \mathbb{R}^3)$ with $1 < p < 3/2$ and $\nu \times \mathbf{u} = 0$ on $\partial\Omega$ in the sense of trace. Then there exists a positive constant C depending only on p and the domain Ω such that*

$$\left\| \frac{\mathbf{u}}{|x|} \right\|_{L^p(\Omega)} \leq C(p, \Omega) \left(\|\operatorname{div} \mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{L^p(\Omega)} \right). \quad (3.9)$$

Proof. Since the domain is in C^2 class, there exist a positive constant ϵ being sufficiently small and a C^2 diffeomorphism that straightens the boundary in the neighborhood

$$\mathcal{N} := \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \epsilon\}$$

of the boundary. The number ϵ depends on the domain.

Case (i): $d := \text{dist}(\mathbf{0}, \partial\Omega) > 0$. Decomposing the integral and then by Minkowski's inequality, we get

$$\left\| \frac{\mathbf{u}}{|x|} \right\|_{L^p(\Omega)} \leq \left\| \frac{\eta_1 \mathbf{u}}{|x|} \right\|_{L^p(\Omega)} + \left\| \frac{(1 - \eta_1) \mathbf{u}}{|x|} \right\|_{L^p(\Omega)}, \quad (3.10)$$

where the function η_1 is defined by

$$0 \leq \eta_1(x) \leq 1, \quad \eta_1(x) = 1 \quad \text{in } B_{d/2}, \quad \text{supp } \eta_1 \subset B_d, \quad |D\eta_1| \leq \frac{4}{d}. \quad (3.11)$$

For the first term in the right side of (3.10), applying the classical Hardy inequality and using the $W^{1,p}$ estimate for vector fields with the tangential components on the boundary vanishing, we have

$$\left\| \frac{\eta_1 \mathbf{u}}{|x|} \right\|_{L^p(\Omega)} \leq C(p, \Omega) \|\nabla(\eta_1 \mathbf{u})\|_{L^p(\Omega)} \leq C(p, \Omega) \left(\|\text{div } \mathbf{u}\|_{L^p(\Omega)} + \|\text{curl } \mathbf{u}\|_{L^p(\Omega)} \right).$$

The estimate on the last term of (3.10) is easy to obtain since $(1 - \eta_1) \mathbf{u} = 0$ in the neighborhood of the origin, and thus

$$\left\| \frac{(1 - \eta_1) \mathbf{u}}{|x|} \right\|_{L^p(\Omega)} \leq C(p, \Omega) \|\mathbf{u}\|_{L^p(\Omega)} \leq C(p, \Omega) \left(\|\text{div } \mathbf{u}\|_{L^p(\Omega)} + \|\text{curl } \mathbf{u}\|_{L^p(\Omega)} \right).$$

In this case, the inequality (3.9) follows immediately.

Case (ii): $\mathbf{0} \in \partial\Omega$. We now consider the problem in the neighborhood \mathcal{N} of $\mathbf{0}$ and take the grid of the curvature lines as the curvilinear coordinate system. Note that the forms of the divergence and the curl are invariant under orthogonal transformations. Therefore, without loss of generality, we assume the unit inward normal vector of $\partial\Omega$ at the point $\mathbf{0}$ is $\mathbf{k} = (0, 0, 1)$. We introduce new variables y_1 and y_2 such that $\mathbf{r}(y_1, y_2)$ represents the portion of $\partial\Omega$ near $\mathbf{0}$ with $\mathbf{r}(0, 0) = \mathbf{0}$ and the y_1 - and y_2 -curves on $\partial\Omega$ are the lines of principle curvatures. By the rotations, we assume one of the principle direction at $\mathbf{0}$ is $\mathbf{e}_1 = \mathbf{i} = (1, 0, 0)$, the other principle direction is $\mathbf{e}_2 = \mathbf{j} = (0, 1, 0)$. We take the diffeomorphism map \mathcal{F} near boundary as follows:

$$x = \mathcal{F}(y) = \mathbf{r}(y_1, y_2) + y_3 \mathbf{n}(y_1, y_2), \quad (3.12)$$

where $\mathbf{n}(y_1, y_2)$ is the unit inner normal vector at the point $\mathbf{r}(y_1, y_2) \in \partial\Omega$. Let

$$G_{ij}(y) = \partial_i \mathcal{F}(y) \cdot \partial_j \mathcal{F}(y), \quad G(y) = G_{11}(y)G_{22}(y).$$

Then we get

$$G_{ii}(y) = 1 + o(|y|). \quad (3.13)$$

Let $\hat{\mathbf{u}}(y)$ be the representation of the vector $\mathbf{u}(x)$ under the new coordinate framework $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, that is

$$\hat{\mathbf{u}}(y) = \hat{u}_1(y) \mathbf{E}_1 + \hat{u}_2(y) \mathbf{E}_2 + \hat{u}_3(y) \mathbf{E}_3 = \mathbf{u}(x).$$

Let $\Omega_t = \Omega \cap B_t(X_0)$ with $B_r = B_r(\mathbf{0})$ the ball center $\mathbf{0}$ and radius r , and assume that \mathcal{F} defined by (3.12) maps some domain $\tilde{\Omega}_\epsilon \subset \mathbb{R}_+^3$ onto a subdomain Ω_ϵ , maps the domain $\tilde{\Omega}_{\epsilon/2} \subset \mathbb{R}_+^3$ onto a subdomain $\Omega_{\epsilon/2}$. We decompose the integral by two parts:

$$\left\| \frac{\mathbf{u}}{|x|} \right\|_{L^p(\Omega)} \leq \left\| \frac{\eta_2 \mathbf{u}}{|x|} \right\|_{L^p(\Omega)} + \left\| \frac{(1 - \eta_2) \mathbf{u}}{|x|} \right\|_{L^p(\Omega)}. \quad (3.14)$$

where $\eta_2(x)$ satisfies

$$0 \leq \eta_2(x) \leq 1, \quad \eta_2(x) = 1 \quad \text{in } \Omega_{\epsilon/2}, \quad \text{supp } \eta_2 \subset \bar{\Omega}_\epsilon, \quad |D\eta_2| \leq \frac{4}{\epsilon}.$$

We extend the vector $\hat{\mathbf{u}}$ on $\tilde{\Omega}_\epsilon$ to the lower half space

$$\tilde{\mathbf{u}}(y) = \begin{cases} (\hat{u}_1(y_1, y_2, y_3), \hat{u}_2(y_1, y_2, y_3), \hat{u}_3(y_1, y_2, y_3)) & \text{if } y \in \tilde{\Omega}_\epsilon \subset \mathbb{R}_+^3, \\ (-\hat{u}_1(y_1, y_2, -y_3), -\hat{u}_2(y_1, y_2, -y_3), \hat{u}_3(y_1, y_2, -y_3)) & \text{if } y \in -\tilde{\Omega}_\epsilon \subset \mathbb{R}_-^3 \end{cases} \quad (3.15)$$

and take the even extension for $\hat{\eta}_2(y) = \eta_2(x)$ on $\tilde{\Omega}_\epsilon$

$$\tilde{\eta}_2(y) = \begin{cases} \hat{\eta}_2(y_1, y_2, y_3) & \text{if } y \in \tilde{\Omega}_\epsilon \subset \mathbb{R}_+^3, \\ \hat{\eta}_2(y_1, y_2, -y_3) & \text{if } y \in -\tilde{\Omega}_\epsilon \subset \mathbb{R}_-^3. \end{cases}$$

Since $\tilde{\eta}_2 \tilde{\mathbf{u}} \in W_0^{1,p}(\mathbb{R}^3)$, the classical Hardy inequality gives that

$$\left\| \frac{\tilde{\eta}_2 \tilde{\mathbf{u}}}{|y|} \right\|_{L^p(\mathbb{R}^3)} \leq C \|D(\tilde{\eta}_2 \tilde{\mathbf{u}})\|_{L^p(\mathbb{R}^3)} \leq C \left(\|\text{div}_y(\tilde{\eta}_2 \tilde{\mathbf{u}})\|_{L^p(\mathbb{R}^3)} + \|\text{curl}_y(\tilde{\eta}_2 \tilde{\mathbf{u}})\|_{L^p(\mathbb{R}^3)} \right). \quad (3.16)$$

The extensions of the vector $\hat{\mathbf{u}}$ and the function $\hat{\eta}_2$ imply that

$$\|\text{div}_y(\tilde{\eta}_2 \tilde{\mathbf{u}})\|_{L^p(\mathbb{R}^3)} \leq 2 \left(\|\text{div}_y \hat{\mathbf{u}}\|_{L^p(\tilde{\Omega}_\epsilon)} + \|(\nabla \hat{\eta}_2) \hat{\mathbf{u}}\|_{L^p(\tilde{\Omega}_\epsilon)} \right). \quad (3.17)$$

The representation of the div operator under the curvilinear coordinate system shows that

$$\|\text{div}_y \hat{\mathbf{u}}\|_{L^p(\tilde{\Omega}_\epsilon)} \leq 4\epsilon \|\nabla \mathbf{u}\|_{L^p(\Omega_\epsilon)} + C(p, \Omega) \left(\|\text{div}_x \mathbf{u}\|_{L^p(\Omega_\epsilon)} + \|\mathbf{u}\|_{L^p(\Omega_\epsilon)} \right). \quad (3.18)$$

The last term of (3.17) can be estimated by

$$\|(\nabla \hat{\eta}_2) \hat{\mathbf{u}}\|_{L^p(\tilde{\Omega}_\epsilon)} \leq C(\epsilon, p, \Omega) \|\mathbf{u}\|_{L^p(\tilde{\Omega}_\epsilon)}.$$

Combing the above three inequalities, we now thus have

$$\begin{aligned} \|\text{div}_y(\tilde{\eta}_2 \tilde{\mathbf{u}})\|_{L^p(\mathbb{R}^3)} &\leq 4\epsilon \|\nabla \mathbf{u}\|_{L^p(\Omega)} + C(p, \epsilon, \Omega) \left(\|\text{div}_x \mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{u}\|_{L^p(\Omega)} \right) \\ &\leq C(p, \Omega) \left(\|\text{div} \mathbf{u}\|_{L^p(\Omega)} + \|\text{curl} \mathbf{u}\|_{L^p(\Omega)} \right), \end{aligned}$$

where we have chosen ϵ being small and let ϵ be fixed. Similarly, we can get the estimate for the curl part

$$\|\text{curl}_y(\tilde{\eta}_2 \tilde{\mathbf{u}})\|_{L^p(\mathbb{R}^3)} \leq C(p, \Omega) \left(\|\text{div} \mathbf{u}\|_{L^p(\Omega)} + \|\text{curl} \mathbf{u}\|_{L^p(\Omega)} \right).$$

By the diffeomorphism mapping (3.12), then using the estimates on the div part and the curl part we have

$$\left\| \frac{\eta_2 \mathbf{u}}{|x|} \right\|_{L^p(\Omega)} \leq C(p) \left\| \frac{\tilde{\eta}_2 \tilde{\mathbf{u}}}{|y|} \right\|_{L^p(\mathbb{R}^3)} \leq C(p, \Omega) \left(\|\operatorname{div} \mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{L^p(\Omega)} \right).$$

At last, combining the estimate on the last term of (3.16)

$$\left\| \frac{(1 - \eta_2) \mathbf{u}}{|x|} \right\|_{L^p(\Omega)} \leq C(\epsilon, p, \Omega) \|\mathbf{u}\|_{L^p(\Omega)} \leq C(p, \Omega) \left(\|\operatorname{div} \mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{L^p(\Omega)} \right),$$

the inequality (3.9) follows and the proof is completed. \square

Remark 3.3. *The conclusion of Theorem 3.2 is still true for the vector fields with the normal components on the boundary vanishing under the assumption that the domain is simply-connected. The difference of the proof is replaced the extension (3.15) by*

$$\tilde{\mathbf{u}}(y) = \begin{cases} (\hat{u}_1(y_1, y_2, y_3), \hat{u}_2(y_1, y_2, y_3), \hat{u}_3(y_1, y_2, y_3)) & \text{if } y \in \tilde{\Omega}_\epsilon \subset \mathbb{R}_+^3, \\ (\hat{u}_1(y_1, y_2, -y_3), \hat{u}_2(y_1, y_2, -y_3), -\hat{u}_3(y_1, y_2, -y_3)) & \text{if } y \in -\tilde{\Omega}_\epsilon \subset \mathbb{R}_-^3. \end{cases}$$

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XINGFEI XIANG: DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI 200092, P.R. CHINA;
E-mail address: xiangxingfei@126.com

ZHIBING ZHANG: DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200241, P.R. CHINA;
E-mail address: zhibingzhang29@126.com